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European Journal of Mechanics B/Fluids 23 (2004) 601–605

The algebraically decaying wall jet

Eugen Magyari *, Bruno Keller

Chair of the Physics of Buildings, Institute of Building Technology, Swiss Federal Institute of Technology (ETH) Zürich, Wolfgang-Pauli-Str.1, CH-8093 Zürich, Switzerland

Received 2 December 2002; accepted 20 October 2003

Available online 9 December 2003

Abstract

The possible existence of a self-similar wall jet over a permeable plane surface in the presence of a (suitable) lateral suction of the fluid is proved. In contrast to its well known exponentially decaying counterpart (the Glauert-jet) formed over an impermeable wall, the suction-controlled wall jet decays algebraically with increasing distance from the wall. However, as the suction becomes vanishingly small, it approaches the profile of the Glauert-jet gradually.

Keywords: Boundary-layers; Wall-jet; Permeable surface; Suction; Algebraic decay; Glauert-jet

1. Introduction

Glauert's approach [1] is the most widely spread description of the self-similar plane wall jet formed over an impermeable resting wall. In the Russian literature the same results are often attributed to Akatnov, [2]. Important results concerning the effects of suction, injection and wall motion (stretching) on the wall jet were reported later by Merkin and Needham [3,4]. They have shown that Glauert's basic equations do not admit (similar) wall jet solutions if:

(i) the wall is resting but permeable and blowing or suction is applied,
(ii) the wall is moving (stretching) but impermeable,
(iii) the wall is (resting or) moving, but a lateral injection of the fluid is applied.

Furthermore, Merkin and Needham [3] have also proved that over a (suitably) stretching permeable wall, a steady boundary layer flow can form if an adequate lateral suction of the fluid is present (see also Magyari and Keller, [5]), such that the corresponding solution admits Glauert's solution as a limiting case. The latter result has also been obtained by a different approach by Magyari and Keller independently, [6].

A careful inspection of the non-existence proof given by Merkin and Needham [3] shows that their conclusion (i) is only legitimate if one assumes that the downstream velocity of the jet decays either exponentially or algebraically, but in any case faster than $\eta^{-2/3}$ as $\eta \to \infty$ (η is the dimensionless similarity variable). If, however, a (similarity maintaining) lateral suction is applied and the algebraic decay $\eta^{-2/3}$ is assumed then, even in the case of a resting wall a further wall jet solution exists. The examination of this algebraically decaying wall jet solution is the main issue of the present paper. For the sake of simplicity it will be referred to hereafter as "a-jet", while its exponentially decaying counterpart, the Glauert-jet, will be referred to as "e-jet".

* Corresponding author.
E-mail address: magyari@hbt.arch.ethz.ch (E. Magyari).

0997-7546/$ – see front matter
doi:10.1016/j.euromechflu.2003.10.003

2. Basic equations and the e-jet

The stream function

$$\psi(x,y)=4x^{1/4}\cdot f(\eta),\quad \eta=x^{-3/4}\cdot y \tag{1}$$

of Glauert's self similar e-jet is the solution of the steady boundary-layer equation

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^3\psi}{\partial y^3} \tag{2}$$

along with the impermeability, the no-slip and the asymptotic conditions

$$\psi(x,0)=0,\quad \frac{\partial\psi}{\partial y}(x,0)=0\quad\text{and}\quad \frac{\partial\psi}{\partial y}\to 0\quad\text{as } y\to\infty, \tag{3}$$

respectively, [1]. Its self-similar part $f(\eta)$ satisfies the ordinary differential equation

$$f'''+ff''+2f'^2=0 \tag{4}$$

along with the boundary conditions

$$f(0)=0,\quad f'(0)=0,\qquad f'(\eta)\to 0\quad\text{as } \eta\to\infty. \tag{5}$$

In the above equations all the quantities $\psi,\ x,\ y$ etc. are nondimensional. The stream function is defined by $u(x,y)=\partial\psi/\partial y,\ v(x,y)=-\partial\psi/\partial x,\ x$ and y are the streamwise and the wall normal coordinates, u and v the corresponding (dimensionless) velocity components,

$$u(x,y)=4x^{-1/2}f'(\eta),\qquad v(x,y)=-x^{-3/4}\cdot\left[f(\eta)-3\eta\cdot f'(\eta)\right] \tag{6}$$

and the primes denote derivatives with respect to η.

The well-known implicit form of the analytical solution of the problem (4), (5) found by Glauert [1] is

$$\eta=3^{1/2}\cdot\arctan\left(\frac{(3f)^{1/2}}{2+f^{1/2}}\right)+\ln\left[\frac{(1+f+f^{1/2})^{1/2}}{1-f^{1/2}}\right]. \tag{7}$$

Since Eq. (4) is invariant under the scaling transformation $\{f\to\lambda\cdot f,\ \eta\to\eta/\lambda\}$ and all the boundary conditions (5) are homogeneous, the solution of the problem (4), (5) is determined up to a normalisation factor only. The Glauert-solution (7) corresponds to the normalisation $f(\infty)=1$ of the stream function which in turn results in the entrainment velocity $v(x,\infty)=-x^{-3/4}$ of the e-jet. The skin friction $f''(0)$, the jet velocity $f'(\eta_*)\equiv f'_{\max}$ and η_*, where η_* is the solution of equation $f''(\eta_*)=0$, are then given by

$$f''(0)=2/9,\quad f'_{\max}=0.315\quad\text{and}\quad \eta_*=2.029 \tag{8}$$

respectively, [1]. The downstream velocity profile $f'(\eta)$ of the e-jet decays asymptotically according to:

$$f'(\eta)\to 2\cdot 3^{1/2}\cdot\exp\left(\frac{\pi}{2\cdot 3^{1/2}}-\eta\right)\quad\text{as } \eta\to\infty. \tag{9}$$

The aim of the present paper is to reconsider the boundary value problem (4), (5) with two essential modifications:

(i) by allowing for a wall transpiration velocity which, in order to maintain similarity is assumed to be of the form $v(x,0)=-x^{-3/4}\cdot f_w$, and
(ii) by allowing for a possible non-exponential asymptotic decay of the dimensionless downstream velocity $f'(\eta)$ for $\eta\to\infty$.

Obviously, all the above Eqs. (1)–(6) are valid also for the a-jet which we are interested in, except for the first boundary condition (5) which we replace by the permeability condition $f(0)=f_w$, where f_w is the suction/injection parameter.

3. The a-jet

We first transcribe Eq. (4) into the equivalent form

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[f^{3/2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{-1/2}f'+\frac{2}{3}f^{3/2}\right)\right]=0 \tag{10}$$

which immediately implies

$$f^{3/2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{-1/2}f'+\frac{2}{3}f^{3/2}\right)=const. \tag{11}$$

Taking into account the actual boundary conditions

$$f(0)=f_w,\quad f'(0)=0,\qquad f'(\eta)\to 0\quad \text{as } \eta\to\infty \tag{12}$$

and putting in Eq. (11) $\eta=0$, one obtains for the integration constant the value $f_w\cdot f''(0)$ where $f''(0)\equiv s$ is the dimensionless skin friction. Thus, Eq. (11) becomes

$$2ff''+2f^2f'-f'^2=2sf_w. \tag{13}$$

Letting here $\eta\to\infty$, one immediately sees that, in a full agreement with the result of Merkin and Needham, [3], the e-jet can only exist if the wall is impermeable $(f_w=0)$. If instead, the algebraic asymptotic behaviour

$$f'(\eta)\to\gamma\cdot\eta^{-2/3}\to 0,\quad \text{i.e.,}\quad f(\eta)\to 3\gamma\cdot\eta^{1/3}\to\infty\quad \text{as } \eta\to\infty \tag{14}$$

is prescribed, where γ is a positive constant, then Eq. (13) can be satisfied for $\eta\to\infty$ with a non-vanishing f_w and

$$s=\frac{9\gamma^3}{f_w}. \tag{15}$$

(The condition $\gamma>0$ results from the physical requirement $f'(\eta)>0$ for $\eta>0$.)

Now, integrating Eq. (4) once from $\eta=0$ to ∞ and taking into account the boundary conditions (12) one obtains for the skin friction the integral formula

$$s=\int_0^\infty f'^2\,\mathrm{d}\eta \tag{16}$$

which holds both for the e-jet and a-jet as well. Therefore, the skin friction s (as expected) is always positive. This, together with $\gamma>0$ requires $f_w>0$ which in turn, according to $v(x,0)=-x^{-3/4}\cdot f_w$ means that the existence of the a-jet with asymptotic behaviour (14) and skin friction (15) requires a (suitable) lateral suction of the fluid.

In this way, for given (positive) values of γ and f_w the problem of the a-jet reduces basically either to the (numerical) solving of the third order equation (4) along with the “initial conditions”

$$f(0)=f_w,\quad f'(0)=0,\quad f''(0)\equiv s=9\gamma^3/f_w \tag{17}$$

or, equivalently to the solving of the second order equation (13) along with the first two initial conditions (17) and $sf_w=9\gamma^3$. These well defined initial value problems guarantee the existence of a unique solution for any given γ and f_w. Thus, for any specified value of $f_w>0$ we obtain a one parameter family of a-jet solutions, the parameter being $\gamma>0$. Obviously, one of the solutions of this family corresponds to the same value of the skin friction $s=2/9$ as the Glauert-jet. It is obtained for

$$\gamma\equiv\gamma_0=\left(\frac{2f_w}{81}\right)^{1/3}. \tag{18}$$

Now, in the limiting case when $f_w\to 0$ and when simultaneously $\gamma\to 0$ according to Eq. (18), the a-jet profile must go over into the e-jet profile. This gradual crossover of the a-jet into the e-jet is shown in Fig. 1. Concerning this crossover, it is also interesting to calculate the momentum flux M and the volume flux Q for the e-jet and the a-jet, respectively. In both of these cases we obtain

$$M=\int_0^\infty u^2\,\mathrm{d}y=\frac{16}{x^{1/4}}\int_0^\infty f'^2\,\mathrm{d}\eta=\frac{16s}{x^{1/4}}, \tag{19}$$

$$Q=\int_0^\infty u\,\mathrm{d}y=4x^{1/4}\int_0^\infty f'\,\mathrm{d}\eta=4x^{1/4}\Big[\lim_{\eta\to\infty}f(\eta)-f_w\Big]. \tag{20}$$

Thus, the momentum flux has the same value for both the jets, but the volume flux is finite only for the e-jet ($Q=4x^{1/4}$) in which case the product $M\cdot Q$ becomes independent of the wall coordinate x. The normalisation $f(\infty)=1$ of the e-jet corresponds to the value $F=M\cdot Q=64s$ of the “Glauert-constant” F. In case of the a-jet, however, as a consequence of its asymptotic behaviour (14), the volume flux (20) becomes infinite, except the above mentioned limiting case $f_w\to 0$

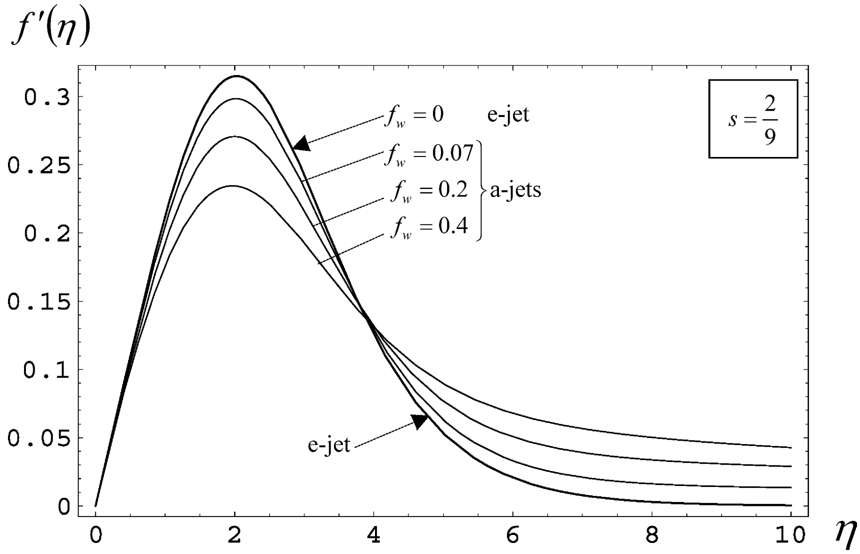

Fig. 1. Gradual crossover of the a-jet into the e-jet for $s = 2/9$ as $f_w \to 0$, while at the same time $\gamma \to 0$ according to Eq. (18).

and $\gamma = \gamma_0 \to 0$, where $f(\eta) - f_w \to 1$ as $\eta \to \infty$. One indeed finds, e.g., $f(20) - f_w = 1.002879$ for $f_w = 10^{-3}$ and $f(20) - f_w = 1.000008$ for $f_w = 10^{-6}$ such that $F \to 64s$ also in this case. Similarly, the entrainment velocity of the a-jet,

$$v(x,\infty) = -x^{-3/4} \cdot \lim_{\eta\to\infty}\left[f(\eta) - 3\eta \cdot f'(\eta)\right] \tag{21}$$

which, as a consequence of the behaviour (14), for finite values of f_w and γ_0 is a vanishing quantity, in the limiting case $f_w \to 0$ and $\gamma = \gamma_0 \to 0$ goes over into the entrainment velocity of the e-jet, $v(x,\infty) = -x^{-3/4}$.

Finally, let us examine the feasibility of the a-jet type boundary layer flows. To this end it is sufficient to inspect the asymptotic behaviour of the downstream velocity

$$u \to 4\gamma y^{-2/3} \quad \text{as } y \to \infty \tag{22}$$

as obtained from the first Eq. (6) by taking into account Eqs. (14) and (1).

Now, Eq. (22) shows that, in contrast to the e-jet formed in a quiescent fluid which in the outer region is considered to be inviscid, the a-jet represents instead a laminar wall-bounded flow driven by an external shear flow of asymptotic velocity $U_\infty = 4\gamma y^{-2/3}$. In other words, the a-jet is nothing than the assimilation of a suction controlled wall jet into an external shear flow. This aspect has also been discussed recently by Magyari et al. [7] in a more general context of the boundary layer flows driven by a power-law shear over permeable plane surfaces. For the case of impermeable ($f_w = 0$) plane surfaces the same problem of the boundary layer flows driven by rotational velocities of the form $U_\infty(y) = \beta y^\alpha$ was investigated comprehensively by Weidman et al. [8]. Weidman et al. have shown [8] that the value $\alpha \equiv \alpha_0 = -2/3$ of the velocity exponent represents the lower limit of existence of such wall-bounded flows, where a singularity occurs. The similarity solution at $\alpha = -2/3$ has an exponential decay in the far field and corresponds to the Glauert-jet, [7]. This result is in full agreement with the findings of the present paper concerning the limiting case $f_w \to 0$ of the family of the a-jet solutions with the same skin friction $s = 2/9$ and momentum flux (19) as the Glauert-jet.

4. Summary and conclusions

The present paper has shown that in addition of the well known exponentially decaying plane wall jet (Glauert-jet, or "e-jet") formed on an impermeable surface, over a permeable plane wall a suction controlled wall jet of the same momentum flux can exist, which, however, decays algebraically ("a-jet") with increasing distance from the wall. In the limiting case of a vanishing suction velocity, the a-jet goes over gradually into the e-jet. It has been argued that the a-jet in fact is nothing than the assimilation of a suction controlled wall jet into an external shear flow of asymptotic velocity $U_\infty = 4\gamma y^{-2/3}$.

It is a remarkable feature of the basic equations of the boundary layer theory that at some limiting values of the parameter involved (as $f_w \to 0$ and $\gamma = \gamma_0 \to 0$ in the present case), exponentially and algebraically decaying velocity profiles match each other. Further examples illustrating this behaviour can also be encountered in the case of the boundary layer flows induced by stretching surfaces [5,6], as well as in the forward [9] and backward free convection boundary layer flows [10] in fluid saturated porous media. In addition to the special case $\alpha = -2/3$ of the boundary layer flows driven by rotational velocities $U_\infty(y) = \beta y^\alpha$ discussed in the present paper, the matching of exponentially and algebraically decaying solutions can also be observed for $\alpha = -1/2$ [7,9]. Finally we mention that the heat transfer characteristics of the wall bounded flows driven by an external shear have not been investigated until now, except for the special case $\alpha = -1/2$ considered in [11]. To attack this problem for the case of the a-jet ($\alpha = -2/3$) represents an open research opportunity.

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